RIGIDITY OF COMMUTING AFFINE ACTIONS ON REFLEXIVE BANACH SPACES

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ABSTRACT. We give a simple argument to show that if α is an affine isometric action of a product $G\times H$ of topological groups on a reflexive Banach space X with linear part π, then either $\pi(H)$ fixes a unit vector or $\alpha|_G$ almost fixes a point on X.

It follows that if π is an isometric linear representation of a topological group G on a reflexive space X such that $\pi(Z(G))$ has no fixed unit vectors, then the reduced cohomology vanishes, i.e., $\overline{H}^1(G,\pi)=0$.

1. LINEAR REPRESENTATIONS AND COCYCLES

When X is a vector space, the group of bijective affine transformations of X, Aff(X), can be decomposed as a semidirect product

$$\mathrm{Aff}(X)=GL(X)\ltimes X,$$

with respect to the natural action of $GL(X)$ on X. The product in $GL(X)\ltimes X$ is then simply $(T,x)\cdot(S,y)=(TS,Ty+x)$, while the corresponding action of $(T,x)\in GL(X)\ltimes X$ on X is given by $(T,x)\cdot y=Ty+x$.

Thus, an action α of a group G by affine transformations of the vector space X can be viewed as a homomorphism of G into Aff(X), which thus can be split into a linear representation $\pi\colon G\to GL(X)$, called the *linear part of* α, and an associated *cocycle* $b\colon G\to X$ such that the following *cocycle identity* holds,

$$b(gf)=\pi(g)b(f)+b(g),$$

for all $g,f\in G$.

If, moreover, X is a reflexive Banach space and $\pi\colon G\to GL(X)$ is a fixed isometric linear representation of a topological group G on X that is *strongly continuous*, i.e., such that for every $x\in X$ the map $g\in G\mapsto gx\in X$ is continuous, we can consider the corresponding vector space $Z^1(G,\pi)$ of continuous cocycles $b\colon G\to X$ associated to π. The subspace $B^1(G,\pi)\subseteq Z^1(G,\pi)$ consisting of those cocycles b for which the corresponding affine action α fixes a point on X, i.e., for which there is some $x\in X$ such that $b(g)=x-\pi(g)x$ for all $g\in G$, is called the set of *coboundaries*. Note that if b is a coboundary, then $b(G)$ is a bounded subset of X. Conversely, if $b(G)$ is a bounded set, then any orbit $\mathcal{O}$ of the corresponding affine action is bounded and so, by reflexivity of X, its closed convex hull $C=\overline{\mathrm{conv}}(\mathcal{O})$ is a weakly compact convex set on which G acts by affine isometries. It follows by the Ryll-Nardzewski fixed point theorem [4] that G fixes a point on C, meaning that b must be a coboundary.

Every compact set $K\subseteq G$ determines a seminorm $\|\cdot\|_K$ on $Z^1(G,\pi)$ by $\|b\|_K=\sup_{g\in K}\|b(g)\|$ and the family of seminorms thus obtained endows $Z^1(G,\pi)$ with a locally convex topology. With this topology, one sees that a cocycle b belongs to the

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closure $\overline{B^1(G,\pi)}$ if and only if the corresponding affine action $\alpha=(\pi,b)$ *almost has fixed points*, that is, if for any compact set $K\subseteq G$ and $\epsilon>0$ there is some $x=x_{K,\epsilon}\in X$ verifying

$$\sup_{g\in K}\|\big(\pi(g)x+b(g)\big)-x\|=\sup_{g\in K}\|b(g)-\big(x-\pi(g)x\big)\|<\epsilon.$$

If, for any K, we can choose $x=x_{K,1}$ above to have arbitrarily large norm, we see that the supremum

$$\sup_{g\in K}\Big\|\pi(g)\frac{x}{\|x\|}-\frac{x}{\|x\|}\Big\|<\frac{\sup_{g\in K}\|b(g)\|+1}{\|x\|}$$

can be made arbitrarily small, which means that the linear action π *almost has invariant unit vectors*. If, on the other hand, for some K the choice of $x_{K,1}$ is bounded (but non-empty), then the same bound holds for any compact $K'\supseteq K$, whereby we find that $b(G)\subseteq X$ is a bounded set, i.e., that $b\in B^1(G,\pi)$. Thus, this shows that if π does not almost have invariant unit vectors, the set $B^1(G,\pi)$ will be closed in $Z^1(G,\pi)$. In fact, if $b\in Z^1(G,\pi)\setminus B^1(G,\pi)$ and π does not almost have invariant unit vectors, then for any constant c there is a compact set $K\subseteq G$ such that no vector is $(\alpha(K),c)$-invariant, where $\alpha=(\pi,b)$.

Conversely, a result of A. Guichardet [3], valid for locally compact σ-compact G, states that if π does not have invariant unit vectors and $B^1(G,\pi)$ is closed in $Z^1(G,\pi)$, then π does not almost have invariant unit vectors.

We define the *first cohomology group* of G with coefficients in π to be the quotient space $H^1(G,\pi)=Z^1(G,\pi)/B^1(G,\pi)$, while the *reduced cohomology group* is $\overline{H^1}(G,\pi)=Z^1(G,\pi)/\overline{B^1(G,\pi)}$.

2. Affine actions of product groups on reflexive spaces

In the following, let X be a reflexive Banach space, G and H be topological groups and π be a strongly continuous linear isometric representation of $G\times H$ on X. We also fix a cocycle $b\in Z^1(G\times H,\pi)$ and let α be the corresponding affine isometric action of $G\times H$ on X.

Proposition 1. *One of the following must hold,*

1. *there is a $\pi(H)$-invariant unit vector,*
2. *for any closed convex $\alpha(H)$-invariant sets $C\subseteq X$, $\alpha|_G$ almost has fixed points on C.*

Proof. Assume that there are no $\pi(H)$-invariant unit vectors in X. Then, if $\pi^n\colon H\to GL(X^n)$ denotes the diagonal representation on $X^n=(X\oplus\ldots\oplus X)_2$, $\pi^n(H)$ has no invariant unit vectors on X^n. By reflexivity, for any $x\in X^n$, $C=\overline{\mathrm{conv}}(\pi^n(H)x)$ is a $\pi^n(H)$-invariant weakly compact convex subset of X^n and thus, by the Ryll-Nardzewski fixed point theorem, $\pi^n(H)$ fixes a point on C, whereby $0\in\overline{\mathrm{conv}}(\pi^n(H)x)$. Therefore, for any $\epsilon>0$ and $(y_1,\ldots,y_n)\in X^n$ there are $h_i\in H$ and $\lambda_i>0$, $\sum_i\lambda_i=1$, such that for all $k=1,\ldots,n$,

$$\Big\|\sum_i\lambda_i\pi(h_i)y_k\Big\|<\epsilon.$$

In particular, if $C\subseteq X$ is a closed convex $\alpha(H)$-invariant set, $\epsilon>0$ and $K\subseteq G$ compact, fix $y\in C$ and find $g_1,\ldots,g_n\in K$ such that $\{\alpha(g_1)y,\ldots,\alpha(g_n)y\}$ is $\frac{\epsilon}{2}$-dense in $\alpha(K)y$. Choose now h_i and λ_i as above such that

$$\Big\|\sum_i\lambda_i\pi(h_i)(y-\alpha(g_k)y)\Big\|<\frac{\epsilon}{2}$$

for all $k = 1,\dots,n$. Thus, if $g \in K$, pick k such that $\|\alpha(g)y - \alpha(g_k)y\| < \frac{\epsilon}{2}$. Then, since $\big\|\sum_i \lambda_i \pi(h_i)\big\| \leqslant 1$,

$$\begin{aligned}
\Big\|\big(\sum_i \lambda_i \alpha(h_i)y\big) - \alpha(g)\big(\sum_i \lambda_i \alpha(h_i)y\big)\Big\| &= \Big\|\big(\sum_i \lambda_i \alpha(h_i)y\big) - \big(\sum_i \lambda_i \alpha(g)\alpha(h_i)y\big)\Big\| \\
&= \Big\|\big(\sum_i \lambda_i \alpha(h_i)y\big) - \big(\sum_i \lambda_i \alpha(h_i)\alpha(g)y\big)\Big\| \\
&= \Big\|\sum_i \lambda_i \big(\alpha(h_i)y - \alpha(h_i)\alpha(g)y\big)\Big\| \\
&= \Big\|\sum_i \lambda_i \big(\pi(h_i)y - \pi(h_i)\alpha(g)y\big)\Big\| \\
&< \Big\|\sum_i \lambda_i \pi(h_i)(y - \alpha(g)y)\Big\| \\
&< \Big\|\sum_i \lambda_i \pi(h_i)(y - \alpha(g_k)y)\Big\| + \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

In other words, the point $\sum_i \lambda_i \alpha(h_i)y \in C$ is $(\alpha(K),\epsilon)$-invariant. □

Corollary 2. *Let G be a topological group and $Z(G)$ its centre. Suppose π is a strongly continuous isometric linear representation on a reflexive Banach space X such that $\pi(Z(G))$ has no fixed unit vectors. Then $\overline{H^1}(G,\pi) = 0$, i.e., any affine isometric action with linear part π almost has fixed points on X.*

Proof. Fix $b \in Z^1(G,\pi)$ and let $\alpha = (\pi, b)$ denote the corresponding affine action. Then α induces an affine action $\tilde\alpha$ of $G \times Z(G)$ on X by letting the first and second coordinate act separately via α. Since the linear part of $\tilde\alpha$ restricted to the second coordinate is simply $\pi|_{Z(G)}$, it follows by Proposition 1 that $\alpha = \tilde\alpha|_G$ almost has invariant vectors. □

Corollary 2 applies in particular to linear representations of an abelian group G without fixed unit vectors. However, we should note that even this special case fails for more general Banach spaces, e.g., for ℓ_1. To see this, let π denote the left regular representation of $\mathbb{Z}$ on $\ell_1(\mathbb{Z})$ and let $b \in Z^1(\mathbb{Z},\pi)$ be given by $b(n) = e_0 + e_1 + \ldots + e_{n-1}$. Then π has no invariant unit vectors. Also, if $x = \sum_{n=-k}^{k} a_n e_n$ is any finitely supported vector, we have

$$\begin{aligned}
\|x - \alpha(1)x\| =& |a_{-k}| + |a_{-k+1} - a_{-k}| + \ldots + |a_{-1} - a_{-2}| + |a_0 - a_{-1} + 1| \\
&+ |a_1 - a_0| + \ldots + |a_k - a_{k-1}| + |a_k| \\
\geqslant& 1.
\end{aligned}$$

So $\|x - \alpha(1)x\| \geqslant 1$ for all $x \in \ell_1(\mathbb{Z})$ and $b \notin \overline{B^1(G,\pi)}$.

Corollary 3. *If $\alpha(G \times H)$ has no fixed point on X and $\pi(G)$ and $\pi(H)$ no invariant unit vectors, then*

(1) *$\alpha|_G$ and $\alpha|_H$ almost have fixed points, and*
(2) *$\pi|_G$ and $\pi|_H$ almost have invariant unit vectors.*

Proof. Item (1) follows directly from Proposition 1, which means that $b|_G \in \overline{B^1(G,\pi|_G)}$ and $b|_H \in \overline{B^1(H,\pi|_H)}$. However, neither $\alpha(G)$ nor $\alpha(H)$ have fixed points, i.e., $b|_G \notin B^1(G,\pi|_G)$ and $b|_H \notin B^1(H,\pi|_H)$. For if, e.g., $\alpha(H)$ fixed a point $x \in X$, then $C = \{x\}$ would be a closed convex $\alpha(H)$-invariant set on which $\alpha|_G$ would have almost fixed points, i.e., x would be fixed by $\alpha(G)$ and so x would be a fixed point for $\alpha(G \times H)$, contradicting our assumptions. Thus, neither $B^1(G,\pi|_G)$ nor $B^1(H,\pi|_H)$ is closed, whereby (2) follows. $\square$

Corollary 4. *Suppose $G = G_1 \times \ldots \times G_n$ is a product of topological groups and $\pi\colon G \to GL(X)$ is a linear isometric representation on a separable reflexive space X. Then X admits a decomposition into $\pi(G)$-invariant linear subspaces $X = V \oplus Y_1 \oplus \ldots \oplus Y_n \oplus W$, such that*

1. *V is the space of $\pi(G)$-invariant vectors,*
2. *any $b \in Z^1(G,\pi^{Y_i})$ factors through a cocycle defined on G_i,*
3. *$Z^1(G,\pi^W) \subseteq \overline{B^1(G_1,\pi^W)} \oplus \ldots \oplus \overline{B^1(G_n,\pi^W)}$,*

where π^W denotes the restriction of π to the invariant subspace W and similarly for Y_i.

Proof. By Theorem 4.10 of [2], for any group of linear isometries of a separable reflexive space Y there is an invariant decomposition of Y into the subspace of fixed points and a canonical complement. Thus, by recursion on the size of $s \subseteq \{1,\ldots,n\}$, we obtain a $\pi(G)$-invariant decomposition

$$X = \sum_{s \subseteq \{1,\ldots,n\}} X_s,$$

where every non-zero $x \in X_s$ is fixed by $\pi\big(\prod_{i \notin s} G_i\big)$ and by none of $\pi(G_i)$ for $i \in s$. So if $b \in Z^1(G,\pi^{X_s})$ and $g \in \prod_{i\notin s} G_i$, then for any $h \in \prod_{i\in s} G_i$,

$$b(h)+b(g) = \pi(g)b(h)+b(g) = b(gh) = b(hg) = \pi(h)b(g)+b(h),$$

i.e., $\pi(h)b(g) = b(g)$, which implies that $b(g) = 0$. It follows that if $s \neq \emptyset$, then any $b \in Z^1(G,\pi^{X_s})$ factors through a cocycle defined on $\prod_{i\in s} G_i$.

Also, if $|s| \geqslant 2$, then by Corollary 3 we see that any $b \in Z^1(G,\pi^{X_s})$ can be written as $b = b_1 \oplus \ldots \oplus b_n$, where $b_i \in \overline{B^1(G_i,\pi^{X_s})}$. Thus, if we set $V = X_\emptyset$, $Y_i = X_{\{i\}}$ and $W = \sum_{|s|\geqslant 2} X_s$, the result follows. $\square$

Proposition 1 was shown by Y. Shalom [5] in the special case of locally compact σ-compact G and H and $X = \mathcal{H}$ a Hilbert space, but by different methods essentially relying on the local compactness of G and H and the euclidean structure of X. This also provided the central lemma for the rigidity results of [5] via the following theorem, whose proof we include for completeness.

Theorem 5 (Shalom [5] for locally compact G and H)**.** *Let $\pi\colon G \times H \to GL(\mathcal{H})$ be a strongly continuous isometric linear representation of a product of topological groups on a Hilbert space $\mathcal{H}$ and assume that neither $\pi(G)$ nor $\pi(H)$ have invariant unit vectors. Then $Z^1(G\times H,\pi) = \overline{B^1(G\times H,\pi)}$ and so $\overline{H^1}(G\times H,\pi) = 0$.*

Proof. Let $b \in Z^1(G\times H,\pi)$ be given with corresponding affine isometric action α and fix compact subsets $K \subseteq G$, $L \subseteq H$ and an $\epsilon > 0$. Then, by Proposition 1, the closed convex $\alpha(G)$-invariant set $C \subseteq \mathcal{H}$ of $(\alpha(L),\epsilon/2)$-invariant points is non-empty. Similarly, there is an $(\alpha(K),\epsilon/2)$-invariant point in $\mathcal{H}$.

Now, by the euclidean structure of $\mathcal{H}$, for any $y \in \mathcal{H}$, there is a unique point $P(y) \in C$ closest to y and, as $\alpha(G)$ acts by isometries on $\mathcal{H}$ leaving C invariant, the map P is $\alpha(G)$-equivariant, i.e., $P(\alpha(g)y) = \alpha(g)P(y)$. Moreover, using the euclidean structure again, P is 1-Lipschitz, whereby

$$\|P(y)-\alpha(g)P(y)\| = \|P(y)-P(\alpha(g)y)\| \leqslant \|y-\alpha(g)y\|,$$

for all $y \in \mathscr{H}$ and $g \in G$. In particular, if $y \in \mathscr{H}$ is $(\alpha(K), \epsilon/2)$-invariant, then $P(y)$ is both $(\alpha(K), \epsilon/2)$ and $(\alpha(L), \epsilon/2)$-invariant, i.e., $P(y)$ is $(\alpha(K \times L), \epsilon)$-invariant. Since K, L and ϵ are arbitrary, we have that $b \in \overline{B^1(G \times H, \pi)}$. □

U. Bader, A. Furman, T. Gelander and N. Monod [1] studied the structure of affine actions of product groups on uniformly convex spaces (a subclass of the reflexive spaces) and in this setting obtained a slightly weaker result than Shalom. Namely, if $\pi: G \times H \to GL(X)$ is a strongly continuous isometric linear representation of a product of topological groups on a uniformly convex space X such that neither $\pi(G)$ nor $\pi(H)$ have invariant unit vectors, then either

(a) π almost has invariant unit vectors, or
(b) $Z^1(G \times H, \pi) = B^1(G \times H, \pi)$.

Proposition 1 is somewhat independent of their statement and shows that one can add that $\alpha|_G$ and $\alpha|_H$ almost have fixed points to (a) above.

References

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